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Diffusive thermal conductivity of the A_1-phase of superfluid ^{3}He at low temperatures

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Abstract

The diffusive thermal conductivity tensor of the A_1-phase of superfluid ^{3}He at low temperatures and melting pressure are calculated beyond the s–p approximation, by using the Boltzmann equation approach. The interaction between normal–normal, normal–Bogoliubov and Bogoliubov–Bogoliubov quasiparticles in the collision integrals are considered for important scattering processes such as binary process. At low temperatures, we show that the scattering between Bogoliubov and normal quasiparticles in binary processes plays an important role in the A_1-phase, and Bogoliubov–Bogoliubov interaction is ignorable.

We show that the two normal and superfluid components take part in elements of the diffusive thermal conductivity tensor differently. We obtain the result that the elements of the diffusive thermal conductivities, K_{xx}, K_{yy} and K_{zz}, are proportional to T^{-1}, and also that the superfluid components of the diffusive thermal conductivity tensor, $K_{xx\uparrow}$ and $K_{zz\uparrow}$, are proportional to T^3 and T, respectively.

1. Introduction

^{3}He has two stable principal superfluid phases, A and B, when there is no magnetic field [1]. If a magnetic field is applied, A stretches down to zero pressure, and a new superfluid phase A_1 appears between the A-phase and the normal liquid ^{3}He. As the field is increased, the B-phase shrinks toward lower T, while A_1 grows [1].

In other words, in the presence of a static magnetic field, the A-phase of liquid ^{3}He splits into two phases, A_1 and A_2, where in the A_1-phase only a single spin population is paired and the A_2-phase contains two independently paired spin populations [2].

Investigation of the coefficients of the diffusive thermal conductivity of the A_1-phase has not yet received attention. Most theoretical efforts have been concentrated on the evaluation of the diffusive thermal conductivity of the B-phase. In [3–5], the Boltzmann equation was solved in the low temperatures limit for the thermal conductivity of the B-phase exactly. It was found that the diffusive thermal conductivity varies with temperature as T^{-1}, the same as in the normal state. In [3], by using an approximate collision integral which gives nearly exact results in the limits $T \to 0$ and $T \to T_c$, this coefficient was calculated for the whole range of temperatures numerically. In [4], by choosing an appropriate trial solution, the Boltzmann equation was solved variationally for the diffusive thermal conductivity of the B-phase. In [5], the transport coefficients of the B-phase, in zero magnetic field, at low temperatures, along with the calculation of transition probabilities in s–p approximation, were investigated. Also in [6] for the A-phase, the diffusive thermal conductivity was calculated, and it was recently used in [7] to investigate the thermal properties of some materials.

In this paper, we report on a study of the thermal conductivity of the A_1-phase of superfluid ^{3}He at low temperatures, melting pressure and high magnetic field up to 15 T, using a Boltzmann equation approach. We assume that the magnetic field is sufficiently high so that all of the quasiparticles with spin up go to the superfluid state and all quasiparticles with spin down stay in the normal state. In other words, the superfluid part in ^{3}He is associated with the Cooper pair condensate and the normal fluid part with the unpaired fermions in excited states. In fact, the high magnetic field is the main reason for the transition to the A_1-phase with an anisotropic gap [1–8]. Also, this field plays an important role in the polarization of spins. Furthermore, the magnetic field explicitly enters into the form of the transition temperature. However, it should be noted that here in our case of study, since we are dealing with the quasiparticle–quasiparticle interactions, then the magnetic field does not play any explicit role. A similar study in which the magnetic field does not appear can be found in [9–11].

We also consider normal–superfluid interactions which come from the scattering between superfluid quasiparticles in the spin-up population, the so-called Bogoliubov quasiparticles, and the normal fluid quasiparticles in the spin-down population. In a normal Fermi liquid at low temperatures the only important collision process is the binary scattering of quasiparticles, but in a superfluid the quasiparticle number is not conserved, and other processes as well as binary processes can occur. So we take into account the following processes: decay processes in which one Bogoliubov quasiparticle decays into three and coalescence processes in which three Bogoliubov quasiparticles coalesce to produce one.

The Pfitzner procedure [12] has been used in the calculation of the quasiparticle scattering amplitude (QSA) which appears in collision integrals of Boltzmann equation. This is the same case for calculating the shear viscosity of the A_1-phase of superfluid ^{3}He at low temperatures and near critical temperature [9–11]. For a system as dense as ^{3}He there is no physical reason to expect the contribution from higher partial waves in the QSA to be negligible, whereas in the Pfitzner procedure, necessary conditions are established to explicitly construct exchange-symmetric scattering amplitudes by adding higher angular momentum components. In this procedure, a general polynomial expansion of the QSA is constructed. This expansion fully contains the s–p approximation as a special case. Therefore the Pfitzner procedure improves on other mentioned approximations.

The paper is organized as follows. In section 2, after introducing and discussing proper transition probabilities for the A_1-phase, the related Boltzmann equation is investigated. In section 3, the diffusive thermal conductivity tensor has been calculated at low temperatures, and in section 4, we give some remarks and concluding results.

2. Transition probabilities and the Boltzmann equation procedure

The interaction term in the Hamiltonian of the A_1-phase of superfluid ^{3}He, H, can be written as [10, 11]

$$
\begin{aligned}
H = \frac{1}{4} \sum_{\vec{P}_1, \vec{P}_2, \vec{P}_3, \vec{P}_4} \big\{ & \big[\langle 3\uparrow 4\uparrow | T | 1\uparrow 2\uparrow \rangle \big(u_{4\uparrow\uparrow} \alpha_{4\uparrow}^{\dagger} - v_{4\uparrow\uparrow}^{*} \alpha_{-4\uparrow} \big) \big(u_{3\uparrow\uparrow} \alpha_{3\uparrow}^{\dagger} - v_{3\uparrow\uparrow}^{*} \alpha_{-3\uparrow} \big) \\
& \times \big(u_{1\uparrow\uparrow} \alpha_{1\uparrow} - v_{1\uparrow\uparrow} \alpha_{-1\uparrow}^{\dagger} \big) \big(u_{2\uparrow\uparrow} \alpha_{2\uparrow} - v_{2\uparrow\uparrow} \alpha_{-2\uparrow}^{\dagger} \big) \big] \\
& + \big[\langle 3\downarrow 4\downarrow | T | 1\downarrow 2\downarrow \rangle a_{4\downarrow}^{\dagger} a_{3\downarrow}^{\dagger} a_{1\downarrow} a_{2\downarrow} \big] \\
& + \big[\langle 3\downarrow 4\uparrow | T | 1\uparrow 2\downarrow \rangle \big(u_{4\uparrow\uparrow} \alpha_{4\uparrow}^{\dagger} - v_{4\uparrow\uparrow}^{*} \alpha_{-4\uparrow} \big) a_{3\downarrow}^{\dagger} \big(u_{1\uparrow\uparrow} \alpha_{1\uparrow} - v_{1\uparrow\uparrow} \alpha_{-1\uparrow}^{\dagger} \big) a_{2\downarrow} \big] \\
& + \big[\langle 3\uparrow 4\downarrow | T | 1\uparrow 2\downarrow \rangle a_{4\downarrow}^{\dagger} \big(u_{3\uparrow\uparrow} \alpha_{3\uparrow}^{\dagger} - v_{3\uparrow\uparrow}^{*} \alpha_{-3\uparrow} \big) \big(u_{1\uparrow\uparrow} \alpha_{1\uparrow} - v_{1\uparrow\uparrow} \alpha_{-1\uparrow}^{\dagger} \big) a_{2\downarrow} \big] \\
& + \big[\langle 3\downarrow 4\uparrow | T | 1\downarrow 2\uparrow \rangle \big(u_{4\uparrow\uparrow} \alpha_{4\uparrow}^{\dagger} - v_{4\uparrow\uparrow}^{*} \alpha_{-4\uparrow} \big) a_{3\downarrow}^{\dagger} a_{1\downarrow} \big(u_{2\uparrow\uparrow} \alpha_{2\uparrow} - v_{2\uparrow\uparrow} \alpha_{-2\uparrow}^{\dagger} \big) \big] \\
& + \big[\langle 3\uparrow 4\downarrow | T | 1\downarrow 2\uparrow \rangle a_{4\downarrow}^{\dagger} \big(u_{3\uparrow\uparrow} \alpha_{3\uparrow}^{\dagger} - v_{3\uparrow\uparrow}^{*} \alpha_{-3\uparrow} \big) a_{1\downarrow} \big(u_{2\uparrow\uparrow} \alpha_{2\uparrow} - v_{2\uparrow\uparrow} \alpha_{-2\uparrow}^{\dagger} \big) \big] \big\}.
\end{aligned} \tag{1}
$$

The first and the second brackets in equation (1), are related to the superfluid and normal component interactions, respectively. The interaction Hamiltonian contains the terms $\alpha_4^{\dagger}\alpha_3^{\dagger}\alpha_{-2}^{\dagger}\alpha_1$, $\alpha_4^{\dagger}\alpha_3^{\dagger}\alpha_2\alpha_1$, $\alpha_4^{\dagger}\alpha_1\alpha_{-3}\alpha_2$, $\alpha_4^{\dagger}\alpha_3^{\dagger}\alpha_{-2}^{\dagger}\alpha_{-1}^{\dagger}$ and $\alpha_{-4}\alpha_{-3}\alpha_2\alpha_1$. These terms decay one quasiparticle into three, convert two quasiparticles into two, coalescence three quasiparticles into one, create four quasiparticles from the condensate and scatter four quasiparticles into the condensate, respectively. The last two processes are not allowed, because in each process the total energy should be conserved. It is noted that for obtaining H the following Bogoliubov transformation has been used for spin up:

$$a_{\vec{p},\uparrow} = u_{\vec{p},\uparrow\sigma'}\alpha_{\vec{p},\sigma'} - v_{\vec{p},\uparrow\sigma'}\alpha_{-\vec{p},\sigma'}^{\dagger} \tag{2}$$

$$a_{\vec{p},\uparrow}^{\dagger} = v_{\vec{p},\uparrow\sigma'}^{*}\alpha_{\vec{p},\sigma'} + u_{\vec{p},\uparrow\sigma'}\alpha_{-\vec{p},\sigma'}^{\dagger} \tag{3}$$

where the matrix elements $u_{\vec{p},\sigma\sigma'}$ and $v_{\vec{p},\sigma\sigma'}$ can be chosen for the A$_1$-phase as [13]

$$u_{\vec{p},\sigma\sigma'} = \left[\frac{1}{2}\left(1+\frac{\varepsilon_{\vec{p}}}{E_{\vec{p}}}\right)\right]^{1/2}\delta_{\sigma\sigma'}; \qquad v_{\vec{p},\sigma\sigma'} = \left[\frac{1}{2}\left(1-\frac{\varepsilon_{\vec{p}}}{E_{\vec{p}}}\right)\right]^{1/2}\delta_{\sigma\sigma'}.$$

In the A$_1$-phase we may write $E_{\vec{p}} = (\varepsilon_{\vec{p}}^2 + |\Delta_{\vec{p}\uparrow\uparrow}|^2)^{1/2}\ \mathrm{sgn}\ \varepsilon_{\vec{p}}$, where $\varepsilon_{\vec{p}}$ is the normal-state quasiparticle energy measured with respect to the chemical potential and $\Delta_{\vec{p}\uparrow\uparrow}$ is the magnitude of the gap in the direction $\vec{p}$ on the Fermi surface [13]. $|\Delta_{\vec{p}\uparrow\uparrow}|$ is equal to $\Delta(T)\sin\theta_{\mathrm{p}}$, where $\Delta(T)$ is the maximum gap and θ_{p} is the angle between the quasiparticle momentum and gap axis $\hat{\ell}$ that is supposed to be in the direction of z-axis. For the non-unitary state of the A$_1$-phase, we have the following properties for u and v [13]:

$$u_{-\vec{p},\uparrow\uparrow} = u_{\vec{p},\uparrow\uparrow}; \qquad v_{-\vec{p},\uparrow\uparrow} = -v_{\vec{p},\uparrow\uparrow}. \tag{4}$$

Because of the appearance of transition probabilities in the collision integrals of Boltzmann equation, it is proper to discuss them now. By using the golden rule, the transition probabilities related to Bogoliubov quasiparticles interaction due to binary, coalescence, and decay processes, respectively, are

$$
\begin{aligned}
W_{22}(\uparrow\uparrow) &= 2\pi|\langle 4\uparrow 3\uparrow|H|1\uparrow 2\uparrow\rangle|^2 \\
W_{31}(\uparrow\uparrow) &= 2\pi|\langle 4\uparrow|H|1\uparrow, 2\uparrow, -3\uparrow\rangle|^2 \\
W_{13}(\uparrow\uparrow) &= 2\pi|\langle 3\uparrow, 4\uparrow, -2\uparrow|H|1\uparrow\rangle|^2
\end{aligned} \tag{5}
$$

(throughout this paper we put $K_{\mathrm{B}} \equiv \hbar \equiv 1$). The transition probabilities related to normal–Bogoliubov component interaction due to binary, coalescence and decay processes, respectively, are

$$
\begin{aligned}
W_{22}(\downarrow\uparrow) &= 2\pi|\langle 4\uparrow 3\downarrow|H|1\downarrow 2\uparrow\rangle|^2 + 2\pi|\langle 4\downarrow 3\uparrow|H|1\downarrow 2\uparrow\rangle|^2 \\
W_{31}(\downarrow\uparrow) &= 2\pi|\langle 1\downarrow|H|-2\downarrow, 3\uparrow, 4\uparrow\rangle|^2 \\
W_{13}(\downarrow\uparrow) &= 2\pi|\langle 3\uparrow, 4\uparrow, -2\downarrow|H|1\downarrow\rangle|^2.
\end{aligned} \tag{6}
$$

Here some points must be noted. The process of decaying a quasiparticle with spin up is forbidden, because in such a case two outgoing quasiparticles out of three must have spin down which, due to the conservation of the number of quasiparticles in normal state, is not allowed. In a similar manner, the coalescence process of the normal–superfluid interaction with a final spin-down state is also impossible. Moreover, the transition probability related to normal component interactions is

$$W_{22}(\downarrow\downarrow) = 2\pi|\langle 4\downarrow\, 3\downarrow|H|1\downarrow\, 2\downarrow\rangle|^2. \tag{7}$$

Now by using Wick's theorem, H given by equation (1), properties of the matrix elements $u_{\vec{p},\sigma\sigma'}$ and $v_{\vec{p},\sigma\sigma'}$ (equation (4)), and the following relations between T-matrix elements and the scattering amplitudes for pairs of quasiparticles in singlet and triplet states, T_s and T_t,[1]

$$\begin{aligned}
&\langle 4\uparrow\, -1\uparrow|T|\, -3\uparrow\, 2\uparrow\rangle \equiv T_{\mathrm{t_{II}}}, \qquad \langle -1\uparrow\, 3\uparrow|T|\, -4\uparrow\, 2\uparrow\rangle \equiv T_{\mathrm{t_{III}}}\\
&\langle 4\downarrow\, 3\uparrow|T|1\uparrow\, 2\downarrow\rangle = \langle 4\uparrow\, 3\downarrow|T|1\downarrow\, 2\uparrow\rangle \equiv \tfrac{1}{2}(-T_{\mathrm{s_I}} + T_{\mathrm{t_I}})\\
&\langle 4\uparrow\, 3\downarrow|T|1\uparrow\, 2\downarrow\rangle = \langle 4\downarrow\, 3\uparrow|T|1\downarrow\, 2\uparrow\rangle \equiv \tfrac{1}{2}(T_{\mathrm{s_I}} + T_{\mathrm{t_I}})\\
&\langle 4\downarrow\, -1\uparrow|T|\, -3\uparrow\, 2\downarrow\rangle = \langle 4\uparrow\, -1\downarrow|T|\, -3\downarrow\, 2\uparrow\rangle \equiv \tfrac{1}{2}(-T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}})\\
&\langle 4\uparrow\, -1\downarrow|T|\, -3\uparrow\, 2\downarrow\rangle = \langle 4\downarrow\, -1\uparrow|T|\, -3\downarrow\, 2\uparrow\rangle \equiv \tfrac{1}{2}(T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}})\\
&\langle -1\downarrow\, 3\uparrow|T|\, -4\uparrow\, 2\downarrow\rangle = \langle -1\uparrow\, 3\downarrow|T|\, -4\downarrow\, 2\uparrow\rangle \equiv \tfrac{1}{2}(-T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}})\\
&\langle -1\uparrow\, 3\downarrow|T|\, -4\uparrow\, 2\downarrow\rangle = \langle -1\downarrow\, 3\uparrow|T|\, -4\downarrow\, 2\uparrow\rangle \equiv \tfrac{1}{2}(T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}})
\end{aligned} \tag{8}$$

(subscripts I, II and III of T_t and T_s identify different binary processes) [11], we can explicitly write the transition probabilities as follows:

$$\begin{aligned}
W_{22}(\uparrow\uparrow) = 2\pi\{&[(|v_1|^2|v_2|^2|v_3|^2|v_4|^2 + |u_1|^2|u_2|^2|u_3|^2|u_4|^2 + 2u_1v_1u_2v_2u_3v_3u_4v_4)|T_{\mathrm{t_I}}|^2]\\
&+ [(|u_1|^2|v_2|^2|u_3|^2|v_4|^2 + |v_1|^2|u_2|^2|v_3|^2|u_4|^2 + 2u_1v_1u_2v_2u_3v_3u_4v_4)|T_{\mathrm{t_{II}}}|^2]\\
&+ [(|v_1|^2|u_3|^2|u_2|^2|v_4|^2 + |v_2|^2|u_4|^2|u_1|^2|v_3|^2 + 2u_1v_1u_2v_2u_3v_3u_4v_4)|T_{\mathrm{t_{III}}}|^2]\\
&- [(|v_1|^2|v_4|^2u_2v_2u_3v_3 + |v_2|^2|v_3|^2u_1v_1u_4v_4 + |u_2|^2|u_3|^2u_1v_1u_4v_4\\
&+ |u_1|^2|u_4|^2u_2v_2u_3v_3)(T^*_{\mathrm{t_I}}T_{\mathrm{t_{III}}} + T_{\mathrm{t_I}}T^*_{\mathrm{t_{III}}})]\\
&- [(|v_4|^2|v_2|^2u_1v_1u_3v_3 + |v_1|^2|v_3|^2u_2v_2u_4v_4 + |u_1|^2|u_3|^2u_2v_2u_4v_4\\
&+ |u_2|^2|u_4|^2u_1v_1u_3v_3)(T^*_{\mathrm{t_I}}T_{\mathrm{t_{II}}} + T_{\mathrm{t_I}}T^*_{\mathrm{t_{II}}})]\\
&+ [(|u_3|^2|v_4|^2u_1v_1u_2v_2 + |v_1|^2|u_2|^2u_3v_3u_4v_4 + |u_1|^2|v_2|^2u_3v_3u_4v_4\\
&+ |v_3|^2|u_4|^2u_1v_1u_2v_2)(T^*_{\mathrm{t_{III}}}T_{\mathrm{t_{II}}} + T_{\mathrm{t_{III}}}T^*_{\mathrm{t_{II}}})]\},\\
W_{31}(\uparrow\uparrow) = 2\pi|&[(v_1^*v_2^*u_3v_4 - u_1u_2v_3^*u_4)T_{\mathrm{t_I}} + (u_1v_2^*v_3^*v_4 - v_1^*u_2u_3u_4)T_{\mathrm{t_{II}}}\\
&+ (v_1^*u_2v_3^*v_4 - u_1v_2^*u_3u_4)T_{\mathrm{t_{III}}}]|^2,\\
W_{13}(\uparrow\uparrow) = 2\pi|&[(v_1^*u_2v_3v_4 - u_1v_2u_3u_4)T_{\mathrm{t_I}} + (v_1^*v_2v_3u_4 - u_1u_2u_3v_4)T_{\mathrm{t_{II}}}\\
&+ (v_1^*v_2u_3v_4 - u_1u_2v_3u_4)T_{\mathrm{t_{III}}}]|^2,\\
W_{22}(\uparrow\downarrow) = (\pi/2)\{&|u_1|^2|u_3|^2|(-T_{\mathrm{s_I}} + T_{\mathrm{t_I}})|^2 + |v_1|^2|v_3|^2|(-T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}})|^2\\
&+ |u_1|^2|u_4|^2|(T_{\mathrm{s_I}} + T_{\mathrm{t_I}})|^2 + |v_1|^2|v_4|^2|(T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}})|^2\\
&- u_1^*v_1^*u_3v_3(-T_{\mathrm{s_I}} + T_{\mathrm{t_I}})^*(-T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}}) - u_1v_1u_3^*v_3^*(-T_{\mathrm{s_I}} + T_{\mathrm{t_I}})(-T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}})^*\\
&- u_1^*v_1^*u_4v_4(T_{\mathrm{s_I}} + T_{\mathrm{t_I}})^*(T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}}) - u_1v_1u_4^*v_4^*(T_{\mathrm{s_I}} + T_{\mathrm{t_I}})(T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}})^*\},\\
W_{31}(\downarrow\uparrow) = (\pi/2)\{&|v_3|^2|u_4|^2|-T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}}|^2 + |u_3|^2|v_4|^2|T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}}|^2\\
&+ u_3v_3u_4^*v_4^*(-T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}})^*(T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}}) + u_3^*v_3^*u_4v_4(-T_{\mathrm{s_{III}}} + T_{\mathrm{t_{III}}})(T_{\mathrm{s_{II}}} + T_{\mathrm{t_{II}}})^*\},
\end{aligned}$$

[1] Properties of T_s and T_t are given in [14, 15].

$$
\begin{aligned}
W_{13}(\downarrow\uparrow) &= (\pi/2)\{|u_3|^2|v_4|^2|T_{s_{II}} + T_{t_{II}}|^2 + |v_3|^2|u_4|^2|-T_{s_{III}} + T_{t_{III}}|^2 \\
&\quad + u_3 v_3 u_4^* v_4^* (T_{s_{II}} + T_{t_{II}})^* (-T_{s_{III}} + T_{t_{III}}) \\
&\quad + u_3^* v_3^* u_4 v_4 (T_{s_{II}} + T_{t_{II}})(-T_{s_{III}} + T_{t_{III}})^*\}, \\
W_{22}(\downarrow\downarrow) &= 2\pi |T_{t_I}|^2.
\end{aligned}
\tag{9}
$$

In addition, $W_{22}(\downarrow\uparrow)$ is obtained from $W_{22}(\uparrow\downarrow)$ simply by the replacements $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

Our previous calculations [10, 11] show that only $W_{22}(\downarrow\downarrow)$, $W_{22}(\uparrow\uparrow)$, and $W_{22}(\uparrow\downarrow)$ are the dominant terms at low temperatures. In fact, at low temperatures, we have $\sin\theta_{p_i} \cong 0$ ($i = 1, 2, 3, 4$) [6]; therefore the explicit form of the remaining transition probabilities shows that they are negligible when $\upsilon \approx 0$. After doing a little algebra, equation (9) at low temperatures gives

$$W_{22}(\downarrow\downarrow) = 2\pi |T_{t_I}|^2 \tag{10}$$

$$W_{22}(\uparrow\uparrow) \cong 2\pi |T_{t_I}|^2 + (\pi/4)(|T_{t_{II}}|^2 + |T_{t_{III}}|^2) \tag{11}$$

$$W_{22}(\uparrow\downarrow) = W_{22}(\downarrow\uparrow) \cong (\pi/2)(|-T_{s_I} + T_{t_I}|^2 + |T_{s_I} + T_{t_I}|^2). \tag{12}$$

Now, T_t and T_s appearing in equations (10)–(12) are given. By using the Pfitzner procedure [12], T_{t_I} and T_{s_I} are given by

$$
\begin{aligned}
N(0)T_{t_I,s_I}(\nu, P) &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} a_{lk}(k+1)^{1/2}(2l+1)^{1/2}(P^2/4 - 1)^l P_l(\nu) P_{k-l}^{(2l+1,0)}(P^2/2 - 1) \\
k &= 0, 1, \ldots; \qquad l = 0, 1, \ldots, k
\end{aligned}
\tag{13}
$$

where the coefficients with l even (odd) belong to the singlet (triplet) part of the QSA. $N(0) = m^* p_F/\pi^2$, $P_l(\nu)$ and $P_n^{(a,b)}(x)$ are the density of states at the Fermi level, the Legendre polynomials and the Jacobi polynomials, respectively. Definitions of P and ν are $P \equiv 2\cos(\theta/2)$ and $\nu \equiv \cos\varphi$, where θ is the angle between the momenta of the incoming particles, namely $\vec{p}_1$ and $\vec{p}_2$, and φ is the angle between the planes spanned by the momentum vectors of the incoming particles and the outgoing particles.

It is noted that at low temperatures θ is small for the scattering of two superfluid incoming quasiparticles, and its maximum value is $\pi T/\Delta(0)$ [6], where $\Delta(0)$ (maximum gap) is equal to $1.77\,T_C$ due to strong coupling effects [16]. To clarify the reason for the smallness of θ, we note the following points. First, at low temperatures, Bogoliubov quasiparticle momentum vectors are located around the nodes of energy gap; consequently, these vectors make small angles around the gap axis. Second, for Bogoliubov quasiparticles, θ follows the same behaviour of θ_p in being small. Based on the above facts, by using equation (13) for different binary processes, when we truncate the summation in equation (13) for $k = 3$ at melting pressure, then T_{sI}, T_{t_I}, $T_{t_{II}}$ and $T_{t_{III}}$ in equations (10)–(12) take the following explicit forms [10, 11]:

$$
\begin{aligned}
N(0)T_{s_I} &= 2.47 + 6.61\cos^2\frac{\theta}{2} + 17.69\cos^4\frac{\theta}{2} - 11.2\cos^6\frac{\theta}{2} \\
&\quad + (3\cos^2\varphi - 1)\sin^4\frac{\theta}{2}\left(3.86 - 6.72\cos^2\frac{\theta}{2}\right) \\
N(0)T_{t_I} &= \sin^2\frac{\theta}{2}\cos\varphi\left[\left(-3.3 + 2.28\cos^2\frac{\theta}{2} - 5.82\cos^4\frac{\theta}{2}\right)\right. \\
&\quad \left. - 0.74\sin^4\frac{\theta}{2}(5\cos^2\varphi - 3)\right] \\
N(0)T_{t_{II}} &= -\left[4.78 + \theta^2\left(0.54\cos^2\frac{\varphi}{2} - 3.05\sin^2\frac{\varphi}{2}\right)\right], \\
N(0)T_{t_{III}} &= \left[4.78 + \theta^2\left(0.54\sin^2\frac{\varphi}{2} - 3.05\cos^2\frac{\varphi}{2}\right)\right].
\end{aligned}
\tag{14}
$$

Now, we are in the position that we should utilize the quantities calculated above. As mentioned earlier, our motive to calculate the transition probabilities has been to use them in the Boltzmann equation of the A_1-phase. By keeping the terms which contribute to the diffusive thermal conductivity, to first order in $\delta\nu_{p,\sigma}$,[2] we have for the initial quasiparticle with spin up (and similarly for spin down) the Boltzmann equation [1, 14, 15]

$$-\frac{\partial \nu^0_\uparrow}{\partial E_{\vec{p},\uparrow}}\frac{\partial E_{p,\uparrow}}{\partial p_k}\frac{E_{\vec{p},\uparrow}}{T}\frac{\partial T}{\partial r_k} = I(\delta\nu_{p,\uparrow}). \tag{15}$$

Also, I, the collision integral, only consists of binary processes; i.e., $I_{22}(\delta\nu_{p,\sigma})$. Considering the nature of the A_1-phase (at low temperatures) and the explicit forms of the transition probabilities for this phase, some points on I are in order. The function W_{22}, present in $I_{22}(\delta\nu_{p,\sigma})$, stands for $(1/4)W_{22}(\uparrow\uparrow)+(1/2)W_{22}(\uparrow\downarrow)$, and $(1/4)W_{22}(\downarrow\downarrow)+(1/2)W_{22}(\downarrow\uparrow)$ for a spin-up/down initial quasiparticle, respectively. For $I_{22}(\delta\nu_{p,\uparrow})$ it is clear from equations (11), (12) and (14) that the contribution of $W_{22}(\uparrow\uparrow)$, due to its dependence on θ, in the collision integral $I_{22}(\delta\nu_{p,\uparrow})$ is proportional to T^2, while $W_{22}(\uparrow\downarrow)$ results in a temperature-independent constant contribution in this integral. Hence, we conclude that $I_{22}(\delta\nu_{p,\uparrow}) = I_{22}(\uparrow\downarrow)$ (where $I_{22}(\uparrow\downarrow)$ is the part of $I_{22}(\delta\nu_{p,\uparrow})$ associated with $W_{22}(\uparrow\downarrow)$). It should be noted that this is not the case for $I_{22}(\delta\nu_{p,\downarrow}) = I_{22}(\downarrow\downarrow) + I_{22}(\downarrow\uparrow)$. Now, the explicit form of the linearized collision terms in the Boltzmann equation can be obtained. For example, for $I_{22}(\uparrow\downarrow)$ we have

$$\begin{aligned} I_{22}(\uparrow\downarrow) = \frac{(m^*)^3T^2}{32\pi^4}\sum_{m=-1}^{1} U_m p_1^{|m|}(\cos\Theta)\mathrm{e}^{im\Phi}\left(\frac{-\partial\nu^0_\uparrow}{\partial E_1}\right)\int \mathrm{d}x\,K(t,x)\int\frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{\cos\frac{\theta}{2}} \\ \times(|-T_{s_I}+T_{t_I}|^2+|T_{s_I}+T_{t_I}|^2)[q(t)+q(-x)p_2(\cos\theta) \\ - q(x)\{p_1(\cos\theta_{13})+p_1(\cos\theta_{14})\}] \end{aligned} \tag{16}$$

where $t\equiv E_1/T$, $x\equiv E_3/T$, $y\equiv E_4/T$ and $K(t,x) = \frac{\mathrm{e}^{-t}+1}{\mathrm{e}^{-x}+1}\frac{x-t}{\mathrm{e}^{(x-t)}-1}$. $q(t)$ is defined by $\delta\nu_{p,\sigma} = -(1/T)\nu^0_{p,\sigma}(1-\nu^0_{p,\sigma})(\partial E_1/\partial P_{1k})_\mu\, q(t)[\partial T/\partial r_k]$. Also, θ_{1i} is the angle between $\vec{p}_1$ and $\vec{p}_i$, and $\theta_{13,14}$ are related to θ and φ with the relation $\cos\theta_{13,14} = \cos^2\frac{\theta}{2}\pm\sin^2\frac{\theta}{2}\cos\varphi$. By substituting equation (16) in (15) and considering $K(t,x) = K(-t,-x)$, we have [17]

$$\int \mathrm{d}x\;K(t,x)\{q_{s_\sigma}-\lambda_{1s_\sigma}q_{s_\sigma}(x)\} = B_\sigma\left\{\frac{\nabla\mu}{\nabla T}+Tt\left(\frac{\partial E^2}{\partial p^2}\left(\frac{\partial p}{\partial E}\right)^2\right)_\mu\right\} \tag{17}$$

$$\int \mathrm{d}x\;K(t,x)\{q_{a_\sigma}(t)-\lambda_{1a_\sigma}q_{a_\sigma}(x)\} = B_\sigma t \tag{18}$$

where q_{s_σ} and q_{a_σ} are the symmetric and antisymmetric parts of $q(t)$, respectively, and σ is simply the spin index. At low temperatures, equation (18) dominates which results in the fact that equation (17) can be ignored [17]. In equation (18), λ_{1a_σ} and B_σ are introduced as

$$\lambda_{1a_\sigma} = \int\frac{\sin(\theta)}{\cos(\frac{\theta}{2})}\,\mathrm{d}\theta\,\mathrm{d}\varphi\,W_{22}\{1+2\cos\theta\}\bigg/\int\frac{\sin\theta}{\cos(\frac{\theta}{2})}W_{22}\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{19}$$

and

$$B_\sigma = \frac{16\pi^5}{m^{*3}T^2}\left[\int\frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{\cos\frac{\theta}{2}}W_{22}\right]^{-1} \tag{20}$$

from which, by using equations (10), (12), and (14), they take the following explicit forms:

[2] The disturbance $\delta\nu_{p,\sigma}$ is defined by $\nu_{p,\sigma} = \nu^0_{p,\sigma}+\delta\nu_{p,\sigma}$, and $\nu_{p,\sigma}$ is the quasiparticle distribution function of superfluid; in addition, $\nu^0_{p,\sigma}$ is $(\exp(E^0_p/T)+1)^{-1}$.

$$\lambda_{1a\uparrow} \cong 1.19, \qquad \lambda_{1a\downarrow} = 1.11 \tag{21}$$

$$B_{1a\uparrow} \cong \frac{\pi^5}{m^{*^3} T^2} \frac{N(0)^2}{186.53} \qquad \text{and} \qquad B_{1a\downarrow} = \frac{\pi^5}{m^{*^3} T^2} \frac{N(0)^2}{195.81}. \tag{22}$$

Following the Sykes *et al* procedure [17], from equation (18) it is obtained that

$$\int_{-\infty}^{\infty} \mathrm{d}t \frac{\mathrm{d}\nu_\sigma^0}{\mathrm{d}t} q_{1a_\sigma}(t)\, t = \frac{-2B_{1\sigma}}{3 - \lambda_{1a_\sigma}} H(\lambda_{1a_\sigma}) \tag{23}$$

where we have $H(\lambda_{1a_\uparrow}) \cong 0.49$ and $H(\lambda_{1a_\downarrow}) \cong 0.5$.[3]

In the next section, by using equations (21)–(23), we will proceed to calculate the diffusive thermal conductivity.

3. Diffusive thermal conductivity

The diffusive thermal conductivity, in general, is a second-rank tensor, which is defined by the relation

$$J_i = -K_{ij} \nabla_j T \tag{24}$$

where J_i, the energy flux, is

$$J_i = \int \mathrm{d}\tau\, E \frac{\partial E}{\partial P_i} \delta\nu_{\mathrm{p},\sigma}. \tag{25}$$

When $\delta\nu_{\mathrm{p},\sigma}$ is inserted in equation (25) and compared with equation (24), we get

$$K_{ij} = -\frac{4P_\mathrm{F}^3}{m^{*^3}(2\pi)^3} T \int \mathrm{d}\Omega_\mathrm{p}\, \hat{p}_i \hat{p}_j \left[\int \mathrm{d}t\, \frac{\partial \nu_\uparrow^0}{\partial t} q_{1a\uparrow}(t) t + \int \mathrm{d}t\, \frac{\partial \nu_\downarrow^0}{\partial t} q_{1a\downarrow}(t) t \right]. \tag{26}$$

By replacing equation (23) in (26), we have

$$K_{ij} = -\frac{4P_\mathrm{F}^3}{m^{*^3}(2\pi)^3} T \int \mathrm{d}\Omega_\mathrm{p}\, \hat{p}_i \hat{p}_j \left(\frac{-2B_\uparrow}{3 - \lambda_{1a\uparrow}} H(\lambda_{1a\uparrow}) + \frac{-2B_\downarrow}{3 - \lambda_{1a\downarrow}} H(\lambda_{1a\downarrow}) \right). \tag{27}$$

After substituting B_σ, λ_{1a_σ}, and $H(\lambda_{1a_\sigma})$, and taking the angular integrations, we have

$$\begin{aligned} &K_{xy} = K_{xz} = K_{yz} = 0, \\ &K_{xx\downarrow} = K_{yy\downarrow} = K_{zz\downarrow} = 0.06 \frac{P_\mathrm{F}^3}{(m^*)^4} N(0)^2 \frac{1}{T}, \\ &K_{xx\uparrow} = K_{yy\uparrow} = \frac{P_\mathrm{F}^3}{(m^*)^4} N_0^2 \left(0.11 \frac{T^3}{T_\mathrm{c}^4} \right), \\ &K_{zz\uparrow} = 0.15 \frac{P_\mathrm{F}^3 N_0^2}{(m^*)^4} \frac{T}{T_\mathrm{c}^2}. \end{aligned} \tag{28}$$

By noting that $K_{xx} = K_{xx\uparrow} + K_{xx\downarrow}$ and $K_{zz} = K_{zz\uparrow} + K_{zz\downarrow}$, finally we have

$$K_{xx} \equiv K_{yy} = K_{zz} = 0.06 \frac{P_\mathrm{F}^3}{(m^*)^4} N_0^2 \frac{1}{T}. \tag{29}$$

Also, the diffusive thermal conductivity tensor for a system with uniaxial symmetry can be written in terms of the components of the symmetry axis $\hat{l}$ with two coefficients $K_\|$ and $K_\perp$

$$K_{ij} = K_\| \hat{l}_i \hat{l}_j + K_\perp (\delta_{ij} - \hat{l}_i \hat{l}_j).$$

By taking the polar axis along $\hat{l}$, we have $K_{||} = K_{zz}$ and $K_\perp = K_{xx} = K_{yy}$. By using values of v_F [18] and m^*/m [19], finally we have

[3] To see the explicit form of $H(\lambda_{1a})$, refer to [17].

$$K_{xx} \equiv K_{yy} = K_{zz} = 6.00 \times 10^{-5} \frac{1}{T} \qquad \text{(in SI units)} \tag{30}$$

where we have used units in which all constants have their real values, such as $\hbar = 6.62 \times 10^{-34}$ (SI).

It must be stressed that here in our calculation we have made an important assumption. We have supposed that the only dominant mechanisms in the calculation of the thermal conductivity of the A_1-phase of superfluid ^{3}He, at low temperatures, are the ones that are elastic. In the calculation of the thermal conductivity it is expected that both elastic and inelastic scattering processes should be considered. In other words, generally elastic scattering may not be the only phenomenon responsible for the thermal conductivity. This study might be a formidable task, because for example one may need to know details of the impurity potentials and the exact nature of the processes responsible for inelasticity. However, it is known that particle–particle scattering is the only important mechanism limiting the diffusive thermal transport in normal Fermi liquid ^{3}He [14] and superfluid ^{3}He [15]. Actually we lack such information on the A_1-phase of superfluid ^{3}He. In the present case, considering our limited knowledge of the existing processes active in A_1 superfluidity, what can be said is to find a witness. In fact, in viscosity calculations of the A_1-phase of superfluid ^{3}He (both at low temperatures [10, 11] and near T_c [9]), ignoring inelastic processes gave rise to a result compatible to the experiment. This strengthens the idea that in our case of study, we also can assume the ignorability of inelastic processes. However, an exhaustive study of this subject is currently not available. By taking the above facts into account we have considered all possible and important elastic interactions of Bogoliubov and normal quasiparticles.

4. Discussion and conclusion

We have calculated the diffusive thermal conductivity tensor of the A_1-phase of superfluid ^{3}He. We have considered the transition probabilities for the cases where both the normal and Bogoliubov quasiparticles are present in decay, coalescence, and binary processes, beyond the s–p approximation. We have used the Pfitzner procedure to obtain singlet and triplet quasiparticle scattering amplitudes which appear in the transition probabilities. Then, by using these probabilities and the result of the Boltzmann equation procedure, we have obtained the diffusive thermal conductivity tensor of the A_1-phase of superfluid ^{3}He.

It is noted that in the calculation of K_{ij} (or equivalently of both $K_\perp$ and $K_\parallel$), interaction between superfluid and normal fluid is of special importance, whereas the contribution from the interaction between Bogoliubov quasiparticles at low temperatures is negligible. This is also the case in the calculation of the shear viscosity tensor of the A_1-phase at low temperatures. To clarify the point, we consider, for example, the values of λ_σ. Both the $\lambda_\downarrow$ and $\lambda_\uparrow$ include normal–Bogoliubov interaction. In the $\lambda_\downarrow$, in addition, we have normal–normal interaction, which is much smaller than normal–Bogoliubov interaction. Then it is concluded that the $\lambda_\downarrow$ and $\lambda_\uparrow$ are not so different.

Also, however, the number of the quasiparticles in superfluid is not fixed (and then some other processes such as decay and coalescence can occur);, at low temperatures it has been shown that just the binary processes are dominant in the calculation of K_{ij}. This is also the case in the calculation of the thermal diffusive coefficient of the A-phase and the shear viscosity tensor of the A_1-phase.

We have found the components of the diffusive thermal conductivity, $K_{xx\uparrow}$ and $K_{zz\uparrow}$, with T^3 and T temperature dependences, respectively. Also the components of $K_\downarrow$ are proportional to T^{-1}. It has been seen that $K_\uparrow$ does not play an important role in the diffusive thermal conductivity components, in comparison with $K_\downarrow$. Therefore, the components of the diffusive thermal conductivity at low temperatures in the A_1-phase of superfluid ^{3}He are proportional to T^{-1}. It should be recalled that the superfluid effect has come into play via the transition probabilities in the calculation of K.

Now, we should compare the thermal conductivities of the A- and A_1-phases of the superfluid. In the A_1-phase, the temperature dependence of the components of the thermal conductivity is related to λ. If λ is considered fixed with temperature (as in the case of the B-phase), then K_{zz} and K_{xx} have temperature dependences as T^{-1} and T, respectively. Otherwise, K_{zz} has a T^{-3} dependence, and K_{xx} has a T^{-1} dependence (as in the case of the B-phase). In contrast to the A-phase, in the A_1-phase, λ_σ does not play this role in the temperature dependence of the thermal conductivity, and all of the components have a T^{-1} dependence, as for the normal component. However, the underlying physics of this special temperature dependence of thermal conductivity of the A_1-phase, in comparison to the other phases, can be related to the existence of the normal part in the A_1-phase and also the dominance of the normal part in competition to the superfluid part of the A_1-phase. The physical reason why the normal part dominates can, in principle, be attributed to the smallness of the phase volume arguments of the superfluid part of the A_1-phase, at low temperatures, and also to the special structure of the order parameter in this phase in comparison to the other phases.

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